Fair Colorful k-Center Clustering

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EPFL

FNSNF

A Technique for Obtaining True Approximations for k-Center with Covering Constraints

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Adam Kurpisz, Rico Zenklusen

ETH zürich

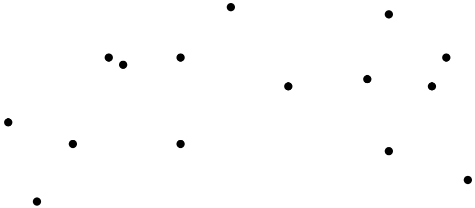
European Research Council
Established by the European Commission

FNSNF

IPCO XXI
2020-06-09

Introduction

k-Center Problem:



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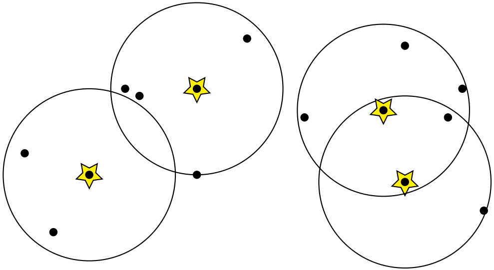
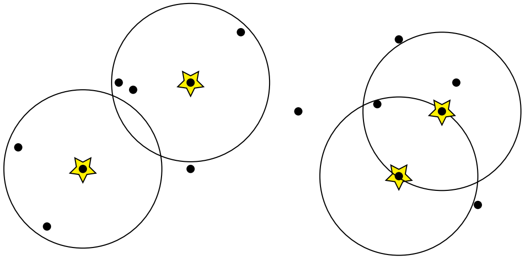


A finite metric space P and integer k, find $C \subseteq P$, $|C| = k$, minimize the maximum distance of $p \in P$ to C.

Introduction

k-Center Problem:

$m = 11$



A finite metric space P and integer k, find $C \subseteq P$, $|C| = k$, minimize the maximum distance of $p \in P$ to C.

- Version with outliers: only m of the points need to be covered

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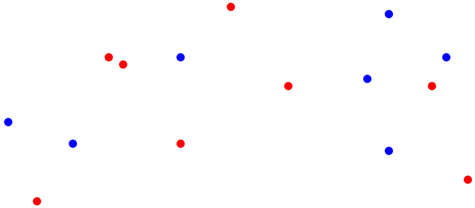
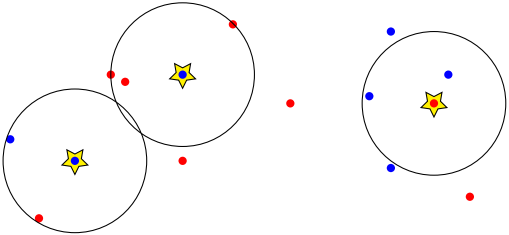
Colorful k-Center Problem:



Colorful k-Center Problem:

$$r = 5$$
$$b = 5$$
$$k = 3$$



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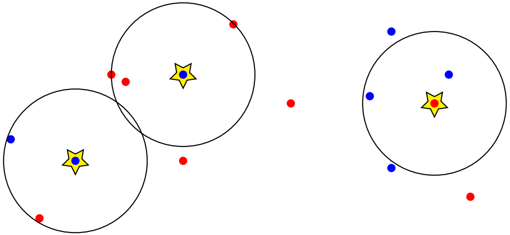

- Points are colored red, R, or blue, B
- Coverage requirements r and b for each color
- Generalizes to more colors
- Better than 2-approx is NP-hard

Motivation

1) **Fairness**

- Each type of element has some service guarantee

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2) Clustering

- Generalizes k-center with outliers problem

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1) Fairness

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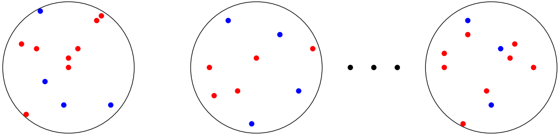
- Generalizes *k*-center with outliers problem
- A novel algorithmic consideration: involves *k*-subset sum problem

$$A = \Sigma_{i=1}^{n} a_i,\ r = k \cdot A + A/2,\ b = k \cdot A - A/2$$



$$r_i = A + a_i,\ b_i = A - a_i$$

Past Work and Results

Problem introduced in [1]

- 2-pseudo approximation opening $(k+1)$ centers
- Constant-factor for 2 colors in $\mathbb{R}^2$

[1] S. Bandyapadhyay, T. Inamdar, S. Pai, K. Varadarajan.: *A Constant Approximation for Colorful k-Center.* In: ESA, pp. 12:1–12:14 (2019).

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Our Results

- 3-approximation for γ number of colors in $|P|^{O(\gamma^2)}$
- Unbounded integrality gap for a linear number of rounds of the Lasserre/Sum-of-Squares hierarchy

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Second Half

- 4-approximation in time $|P|^{O(\gamma)}$
- Assuming $P \neq NP$, inapproximable for unbounded γ
- Assuming ETH, inapproximable for $\gamma = \omega(\log |P|)$

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Challenges

Standard linear programming relaxation has unbounded integrality gap after a linear number of rounds of the Lasserre/Sum-of-Squares hierarchy [2].

$$\sum_{i=1}^{n} 2y_i = k, \quad k \in \mathbb{Z} \text{ odd}$$

$$y_i \in \{0, 1\}$$

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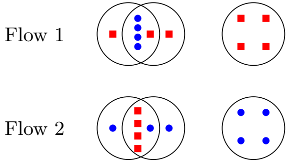


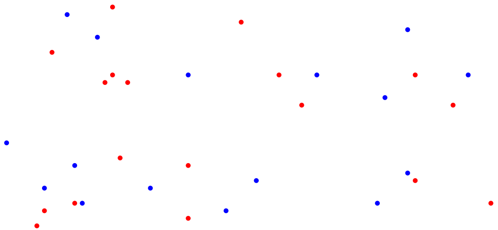
Adding flow-based inequalities to solve a subset-sum problem also does not work.

[2] Grigoriev, D.: Complexity of Positivstellensatz Proofs for the Knapsack. Comput. Complex. 10(2), 139–154 (2001).

Main Ideas

$(k+1)$-pseudo approx. algorithm[1]:

$k = 4$



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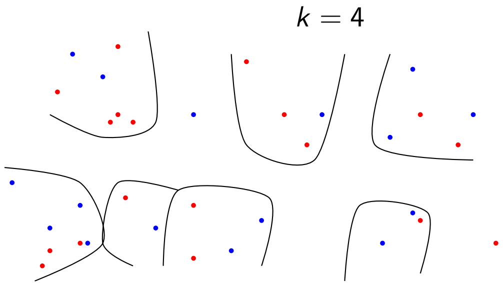

- Use feasible point to natural LP relaxation to get clustering

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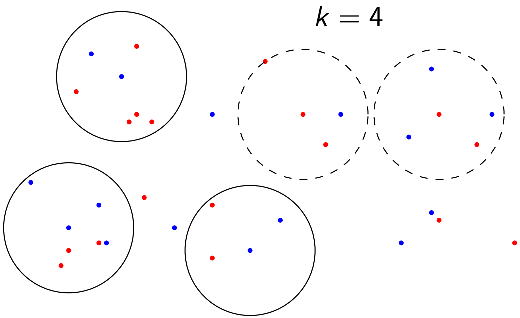

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- Extreme point to simplified LP has at most 2 strictly fractional variables

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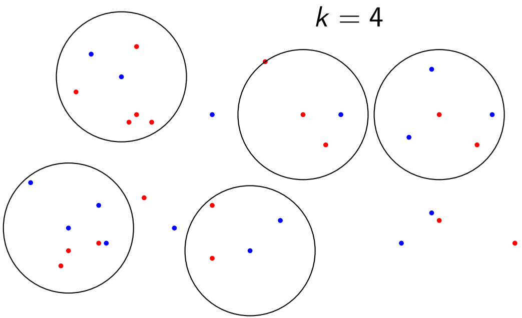

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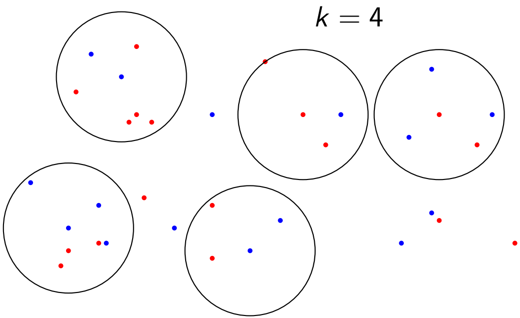

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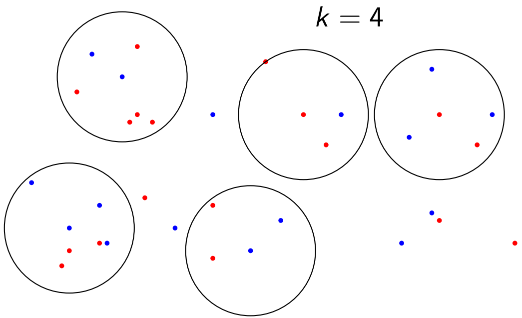

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Idea: Expand balls to cover enough red points, to close fractional center with fewer blue points.

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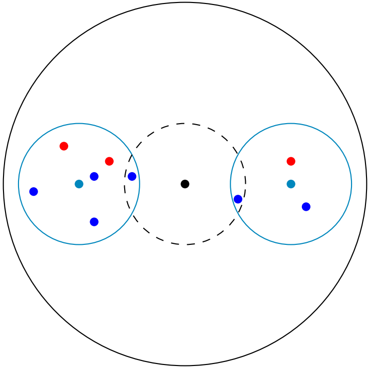
Main Ideas

Intuition:

- OPT solution not *well-separated*: easy



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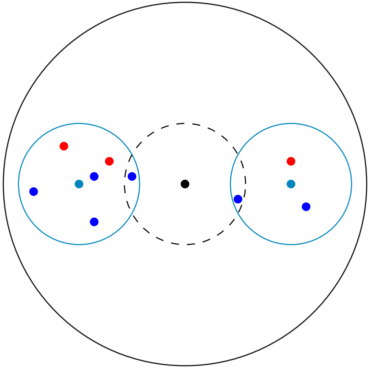
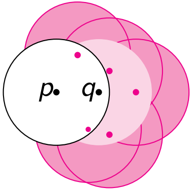


- **OPT** solution well-separated: subset-sum problem

Well-Separated Instances

Phase I: Guess points in an OPT solution, that maximizes the number of red points in a *flower* structure.



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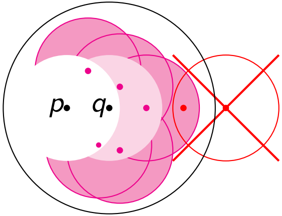

Key Point:

- Separatedness ensures that no ball of OPT intersects expanded region

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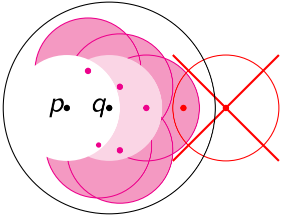

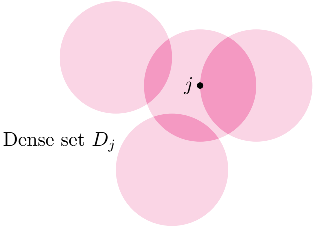

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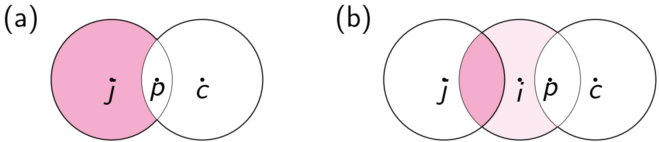
Phase II:

- Remove subsets of remaining points that are *dense*
- Run pseudo-approx. on remaining *sparse* points

Phase II

Required: Removed sets shouldn't "interact" with balls in OPT



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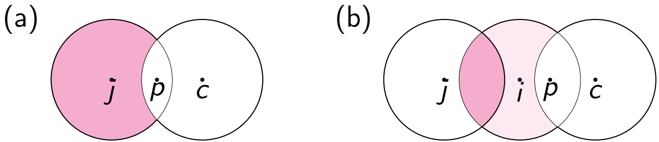

Dynamic program on sets D_j:

- Guess number of red and blue points covered by D_j's, and verify with dynamic program

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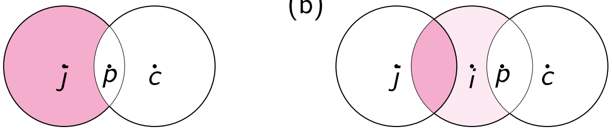
(a) (b)



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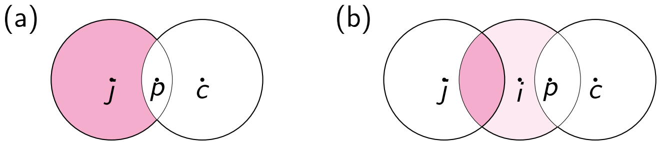

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$\Rightarrow$ covered all required points to get poly-time 2-approximation for well-separated instances!

Conclusion Part I

Result:

- Easily generalizes to γ color classes

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Open Questions:

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And now...